Math Virtual Learning

Calculus AB

Application of Integrals

May 15, 2020

Calculus AB
Lesson: May 15, 2020

Objective/Learning Target:
Lesson 4 Integrals Review
Students will use integrals to solve application problems.

Warm-Up:

Note: This is a review. For more examples refer back to your notes.

Watch Videos: Analyzing Problems
Worked Example

Read Article: Analyzing Problems

Notes:

Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Examples:

At different altitudes in Earth's atmosphere, sound travels at different speeds. The speed of sound $s(x)$ (in meters per second) can be modeled by

$$s(x) = \begin{cases} -4x + 341, & 0 \le x < 11.5 \\ 295, & 11.5 \le x < 22 \\ \frac{3}{4}x + 278.5, & 22 \le x < 32 \\ \frac{3}{2}x + 254.5, & 32 \le x < 50 \\ -\frac{3}{2}x + 404.5, & 50 \le x \le 80 \end{cases}$$

where x is the altitude in kilometers (see Figure 4.34). What is the average speed of sound over the interval $[0, 80]$?

Solution Begin by integrating $s(x)$ over the interval $[0, 80]$. To do this, you can break the integral into five parts.

$$\int_0^{11.5} s(x)\,dx = \int_0^{11.5} (-4x + 341)\,dx = \Big[-2x^2 + 341x\Big]_0^{11.5} = 3657$$

$$\int_{11.5}^{22} s(x)\,dx = \int_{11.5}^{22} (295)\,dx = \Big[295x\Big]_{11.5}^{22} = 3097.5$$

$$\int_{22}^{32} s(x)\,dx = \int_{22}^{32} \left(\frac{3}{4}x + 278.5\right)dx = \Big[\frac{3}{8}x^2 + 278.5x\Big]_{22}^{32} = 2987.5$$

$$\int_{32}^{50} s(x)\,dx = \int_{32}^{50} \left(\frac{3}{2}x + 254.5\right)dx = \Big[\frac{3}{4}x^2 + 254.5x\Big]_{32}^{50} = 5688$$

$$\int_{50}^{80} s(x)\,dx = \int_{50}^{80} \left(-\frac{3}{2}x + 404.5\right)dx = \Big[-\frac{3}{4}x^2 + 404.5x\Big]_{50}^{80} = 9210$$

By adding the values of the five integrals, you have

$$\int_0^{80} s(x)\,dx = 24{,}640.$$

So, the average speed of sound from an altitude of 0 kilometers to an altitude of 80 kilometers is

$$\text{Average speed} = \frac{1}{80}\int_0^{80} s(x)\,dx = \frac{24{,}640}{80} = 308 \text{ meters per second.}$$

Examples:

Example 9.2.2 The acceleration of an object is given by $a(t) = \cos(\pi t)$, and its velocity at time $t = 0$ is $1/(2\pi)$. Find both the net and the total distance traveled in the first 1.5 seconds.

We compute

$$v(t) = v(0) + \int_0^t \cos(\pi u)du = \frac{1}{2\pi} + \frac{1}{\pi}\sin(\pi u)\Big|_0^t = \frac{1}{\pi}(\frac{1}{2} + \sin(\pi t)).$$

The {\it net} distance traveled is then

$$\begin{aligned} s(3/2) - s(0) &= \int_0^{3/2} \frac{1}{\pi}\left(\frac{1}{2} + \sin(\pi t)\right)\,dt \\ &= \frac{1}{\pi}\left(\frac{t}{2} - \frac{1}{\pi}\cos(\pi t)\right)\Big|_0^{3/2} = \frac{3}{4\pi} + \frac{1}{\pi^2} \approx 0.340 \text{ meters.} \end{aligned}$$

To find the {\it total} distance traveled, we need to know when $(0.5 + \sin(\pi t))$ is positive and when it is negative. This function is 0 when $\sin(\pi t)$ is -0.5, i.e., when $\pi t = 7\pi/6$, $11\pi/6$, etc. The value $\pi t = 7\pi/6$, i.e., $t = 7/6$, is the only value in the range $0 \le t \le 1.5$. Since $v(t) > 0$ for $t < 7/6$ and $v(t) < 0$ for $t > 7/6$, the total distance traveled is

$$\begin{aligned} &\int_0^{7/6} \frac{1}{\pi}\left(\frac{1}{2} + \sin(\pi t)\right)\,dt + \left|\int_{7/6}^{3/2} \frac{1}{\pi}\left(\frac{1}{2} + \sin(\pi t)\right)\,dt\right| \\ &= \frac{1}{\pi}\left(\frac{7}{12} + \frac{1}{\pi}\cos(7\pi/6) + \frac{1}{\pi}\right) + \frac{1}{\pi}\left|\frac{3}{4} - \frac{7}{12} + \frac{1}{\pi}\cos(7\pi/6)\right| \\ &= \frac{1}{\pi}\left(\frac{7}{12} + \frac{1}{\pi}\frac{\sqrt{3}}{2} + \frac{1}{\pi}\right) + \frac{1}{\pi}\left|\frac{3}{4} - \frac{7}{12} + \frac{1}{\pi}\frac{\sqrt{3}}{2}.\right| \approx 0.409 \text{ meters.} \end{aligned}$$

Examples:

Example 3: A missile is accelerating at a rate of 4 t m/sec ^{2} from a position at rest in a silo 35 m below ground level. How high above the ground will it be after 6 seconds?

From the given conditions, you find that $a(t) = 4t$ m/sec ^{2}, $v_0 = 0$ m/sec because it begins at rest, and $s_0 = -35$ m because the missile is below ground level; hence,

$$v(t) = \int 4t\,dt = 2t^2$$

and

$$s(t) = \int 2t^2\,dt = \frac{2}{3}t^3 - 35$$

After 6 seconds, you find that $s(6) = \frac{2}{3}(6)^3 - 35\text{m} = 109\text{m}$

hence, the missile will be 109 m above the ground after 6 seconds.

Practice:

Note: You will want to use your calculator to compute the value of the integral [Math-->9:fnInt( ]

1) _____84) A pizza, heated to a temperature of 350 degrees Fahrenheit (°F), is taken out of an oven and placed in a 75°F room at time $t = 0$ minutes. The temperature of the pizza is changing at a rate of $-110e^{-0.4t}$ degrees Fahrenheit per minute. To the nearest degree, what is the temperature of the pizza at time $t = 5$ minutes?

A) 112°F
B) 119°F
C) 147°F
D) 238°F
E) 335°F

2) A spherical tank contains 81.637 gallons of water at time $t = 0$ minutes. For the next 6 minutes, water flows out of the tank at the rate of $9\sin\left(\sqrt{t+1}\right)$ gallons per minute. How many gallons of water are in the tank at the end of the 6 minutes?

(A) 36.606 (B) 45.031 (C) 68.858 (D) 77.355 (E) 126.668

Answer Key:

Once you have completed the problems, check your answers here.

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$x(5) = x(0) + \int_0^5 -110e^{-0.4t}\,dt$

Answer Key:

Once you have completed the problems, check your answers here.

2)

A spherical tank contains 81.637 gallons of water at time $t = 0$ minutes. For the next 6 minutes, water flows out of the tank at the rate of $9\sin\left(\sqrt{t+1}\right)$ gallons per minute. How many gallons of water are in the tank at the end of the 6 minutes?

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$$v(6) = v(0) + \int_0^6 9\sin\left(\sqrt{t+1}\right)dt$$

Additional Practice:

Interactive Practice

More Interactive Practice

Extra Practice with Answers